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# Bright and dark soliton solutions to coupled nonlinear Schrödinger equations

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**Abstract.** A set of two coupled nonlinear Schrödinger equations (CNLS) describes the two mode propagation in an optical fibre. The Painlevé singularity structure analysis singles out two integrable parametric choices, including the Manakov model. Using the results of Painlevé analysis, we succeed in Hirota bilinearizing the CNLS equations in both the anomalous as well as normal dispersion regions for the integrable cases. Solving the Hirota bilinear equations, bright and dark $N$-soliton solutions are explicitly obtained.

## 1. Introduction

The interaction of two optical modes $q_1$ and $q_2$ of shorter wavelengths or longer characteristic lengths of the envelope in a fibre is governed by a system of CNLS equations [1–9] written in dimensionless form as

$$\mathrm{i}q_{1x}+c_1q_{1tt}+2(\alpha|q_1|^2+\beta|q_2|^2)q_1=0 \tag{1a}$$

$$\mathrm{i}q_{2x}+c_2q_{2tt}+2(\beta|q_1|^2+\gamma|q_2|^2)q_2=0 \tag{1b}$$

(and their complex conjugates), where $c_1$, $c_2$, $\alpha$, $\beta$ and $\gamma$ are real parameters. The variables $x$ and $t$ are the normalized distance and time, the parameter $\beta$ is the cross-coupling coefficient and the signum functions $c_1$ and $c_2$ depend on the signs of the group velocity dispersion (GVD) in each mode, that is $c=+1$ for anomalous GVD and $c=-1$ for normal GVD (without loss of generality).

Sahadevan *et al* [10] established that the system (1) possesses Painlevé (P) property for the two specific parametric choices

$$c_1=c_2 \qquad \alpha=\beta=\gamma \tag{2}$$

$$c_1=-c_2 \qquad \alpha=-\beta=\gamma \tag{3}$$

and are thus integrable in these cases. In fact, Zakharov and Schulman [11] earlier established the existance of 'motion invariants' for these cases. The P-property for the system (1) for choices (2) and (3) implies that the solutions of the system (1) must be single-valued about the non-characteristic movable singular manifold $\phi(x, t)$ so that locally the Laurent expansion can be given by

$$q_i(x,t)=\phi^{-1}\sum_{j=0}^{\infty} q_{ij}(x,t)\phi^j(x,t) \qquad i=1,2 \tag{4}$$

(and similar but independent expressions for $q_i^*(x,t)$), where $q_{ij}(x,t)$ are analytic functions of $x$ and $t$. The detailed analysis shows that the Laurent series for $q_1$, $q_2$, $q_1^*$ and $q_2^*$ contain eight arbitrary functions, including the singular manifold $\phi(x,t)$, without the introduction of movable critical manifolds as expected from the nature of equations (1), for the above two specific choices (2) and (3). The Laurent expansion can also be profitably used to construct the Bäcklund transformation (BT) and Hirota bilinearization [12, 13]. In this paper we explicitly obtain the bright and dark $N$-soliton solutions for the choice (2) of the system (1). The bright one-soliton form agrees with the result derivable from the inverse scattering method [3] and a special case of the dark one-soliton agrees with the form of the solution obtained using an ansatz by Kivshar and Turitsyn [8], while that of the remaining higher-order bright and dark $N$-solitons are reported for the first time using the relation between P-analysis and the Hirota technique. Moreover, the solutions for the choice (3) of the equations (1) are also found for the first time.

## 2. Bright solitons

The system (1) under the parametric restriction (2) is nothing but the well known integrable model proposed by Manakov [14]. Kaup and Malomed [3] have recently briefly discussed its role in nonlinear optics using its one-soliton solution obtained from the inverse scattering method. They have pointed out that the Manakov model, besides the birefringence property, covers many other physical phenomena such as soliton trapping and daughter wave ('shadow') formation in optical fibres. The parametric choice (2), namely $\alpha=\beta=\gamma=\mu$ (say), $c_1=c_2=+1$, corresponds to the anomalous GVD region, where bright solitons can exist, so that the system (1) becomes

$$\mathrm{i}q_{1x}+q_{1tt}+2\mu(|q_1|^2+|q_2|^2)q_1=0 \tag{5a}$$

$$\mathrm{i}q_{2x}+q_{2tt}+2\mu(|q_1|^2+|q_2|^2)q_2=0. \tag{5b}$$

Now by truncating the Laurent expansions (4) up to the constant-level term, that is, $q_{ij}=0$, $j\geqslant 2$, we can formally write the BT as

$$q_i=q_{i0}\phi^{-1}+q_{i1} \qquad i=1,2 \tag{6}$$

where the pairs $(q_i, q_{i1})$ are solutions of (5). In order to construct the Hirota bilinear form [15, 16], we consider the vacuum solutions $q_{i1}=0$ $(i=1,2)$ in (6). Then we have

$$q_1=q_{10}\phi^{-1} \qquad q_2=q_{20}\phi^{-1}. \tag{7}$$

This suggests that we take the Hirota bilinear transformation in the form

$$q_1=\frac{g}{f} \qquad q_2=\frac{h}{f} \tag{8}$$

where $g(x,t)$, $h(x,t)$ are complex functions and $f(x,t)$ is a real function.

Substituting (8) into (5), we obtain

$$f[(\mathrm{iD}_x+\mathrm{D}_t^2)g\cdot f]-g[\mathrm{D}_t^2 f\cdot f-2\mu(gg^*+hh^*)]=0 \tag{9a}$$

$$f[(\mathrm{iD}_x+\mathrm{D}_t^2)h\cdot f]-h[\mathrm{D}_t^2 f\cdot f-2\mu(gg^*+hh^*)]=0 \tag{9b}$$

where the Hirota bilinear operators $D_x$ and $D_t^2$ are defined as

$$\mathrm{D}_x^m\mathrm{D}_t^n g(x,t)\cdot f(x,t)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n g(x,t)f(x',t')|_{x=x',t=t'}. \tag{10}$$

Equations (9) can be decoupled as

$$\mathscr{A}_1 g\cdot f=0 \qquad \mathscr{A}_1 h\cdot f=0 \qquad \mathscr{A}_2 f\cdot f=gg^*+hh^* \tag{11}$$

where the bilinear operators $\mathscr{A}_1$ and $\mathscr{A}_2$ are defined as

$$\mathscr{A}_1=(\mathrm{iD}_x+\mathrm{D}_t^2) \qquad \mathscr{A}_2=\frac{1}{2\mu}\mathrm{D}_t^2. \tag{12}$$

In order to obtain the bright soliton solutions, we proceed in the standard way. For example, in order to find the one soliton solution, we assume

$$g=\chi g_1 \qquad h=\chi h_1 \qquad f=1+\chi^2 f_2 \tag{13}$$

where $\chi$ is an arbitrary parameter. Substituting (13) into (12) and then collecting the terms with same power in $\chi$, we obtain:

$$\chi:\qquad \mathscr{A}_1 g_1\cdot 1=0 \qquad \mathscr{A}_1 h_1\cdot 1=0 \tag{14}$$

$$\chi^2:\qquad \mathscr{A}_2(1\cdot f_2+f_2\cdot 1)=g_1g_1^*+h_1h_1^* \tag{15}$$

$$\chi^3:\qquad \mathscr{A}_1 g_1\cdot f_2=0 \qquad \mathscr{A}_1 h_1\cdot f_2=0 \tag{16}$$

and

$$\chi^4:\qquad \mathscr{A}_2 f_2\cdot f_2=0. \tag{17}$$

One can easily check that the solution, which is consistent with the system (14–17), is

$$\begin{gathered} g_1=\exp(\eta_1) \qquad h_1=\exp(\eta_1+\varepsilon) \\ f_2=\frac{\mu(1+\exp(\varepsilon+\varepsilon^*))}{(k_1+k_1^*)^2}\exp(\eta_1+\eta_1^*) \end{gathered} \tag{18}$$

where

$$\eta_1=k_1(t+\mathrm{i}k_1x)+\eta_1^{(0)} \tag{19}$$

and in which $k_1$, $\eta_1^{(0)}$ and $\varepsilon$ are all complex constants in general and the symbol * indicates complex conjugate. Using (18) in (13), after absorbing $\chi$ the bright one-soliton solution can easily be worked out to be

$$q_1=\frac{\varepsilon_1 k_{1\mathrm{R}}\exp\{\mathrm{i}[(k_{1\mathrm{R}}^2+k_{1\mathrm{I}}^2)x+k_{1\mathrm{I}}(t-2k_{1\mathrm{I}}x)+\eta_{1\mathrm{I}}^{(0)}]\}}{\cosh[k_{1\mathrm{R}}(t-2k_{1\mathrm{I}}x)+\zeta]} \tag{20}$$

$$q_2=\frac{\varepsilon_2 k_{1\mathrm{R}}\exp\{\mathrm{i}[(k_{1\mathrm{R}}^2+k_{1\mathrm{I}}^2)x+k_{1\mathrm{I}}(t-2k_{1\mathrm{I}}x)+\eta_{1\mathrm{I}}^{(0)}]\}}{\cosh[k_{1\mathrm{R}}(t-2k_{1\mathrm{I}}x)+\zeta]} \tag{21}$$

where

$$\zeta=\eta_{1R}^{(0)}+\tfrac{1}{2}\ln\left[\frac{\mu(1+\exp(\varepsilon+\varepsilon^*))}{(k_1+k_1^*)^2}\right]$$

$$\varepsilon_1=\left[\frac{1}{\mu(1+\exp(\varepsilon+\varepsilon^*))}\right]^{1/2}\qquad \varepsilon_2=\left[\frac{\exp(2\varepsilon)}{\mu(1+\exp(\varepsilon+\varepsilon^*))}\right]^{1/2}. \tag{22a–c}$$

In the above, the suffices R and I denote the real and imaginary parts. We can also easily check from (22*b*–*c*) that $|\varepsilon_1|^2+|\varepsilon_2|^2=(1/\mu)$. The one-soliton solutions (20)–(22) obtained here by the Hirota method exhibits exactly the same form as the soliton solution reported from the inverse scattering method [3].

Next, in order to find the two-soliton solutions, we can assume

$$g=\chi g_1+\chi^3 g_3\qquad h=\chi h_1+\chi^3 h_3\qquad f=1+\chi^2 f_2+\chi^4 f_4 \tag{23}$$

and proceed as before to obtain

$$g_1=\exp(\eta_1)+\exp(\eta_2)\qquad h_1=\exp(\varepsilon)[\exp(\eta_1)+\exp(\eta_2)]$$

$$f_2=a(1,1^*)\exp(\eta_1+\eta_1^*)+a(1,2^*)\exp(\eta_1+\eta_2^*)$$
$$+a(2,1^*)\exp(\eta_2+\eta_1^*)+a(2,2^*)\exp(\eta_2+\eta_2^*)$$

$$g_3=a(1,2,1^*)\exp(\eta_1+\eta_2+\eta_1^*)+a(1,2,2^*)\exp(\eta_1+\eta_2+\eta_2^*)]$$

$$h_3=\exp(\varepsilon)[a(1,2,1^*)\exp(\eta_1+\eta_2+\eta_1^*)+a(1,2,2^*)\exp(\eta_1+\eta_2+\eta_2^*)]$$

$$f_4=a(1,2,1^*,2^*)\exp(\eta_1+\eta_2+\eta_1^*+\eta_2^*) \tag{24a–f}$$

where

$$\eta_j=k_j(t+ik_jx)+\eta_j^{(0)}\qquad j=1,2 \tag{25}$$

$$a(i,j^*)=\frac{\mu(1+\exp(\varepsilon+\varepsilon^*))}{(k_i+k_j^*)^2}$$
$$a(i,j)=\frac{(k_i-k_j)^2}{\mu(1+\exp(\varepsilon+\varepsilon^*))}\qquad a(i^*,j^*)=\frac{(k_i^*-k_j^*)^2}{\mu(1+\exp(\varepsilon+\varepsilon^*))} \tag{26}$$

$$a(i,j,k^*)=a(i,j)a(i,k^*)a(j,k^*) \tag{27}$$

and

$$a(i,j,k^*,l^*)=a(i,j)a(i,k^*)a(i,l^*)a(j,k^*)a(j,l^*)a(k^*,l^*). \tag{28}$$

Here $k_j$, $\eta_j^{(0)}$ and $\varepsilon$ are all complex constants. Using (24)–(28) in (23) and then in (8), the two-soliton solutions of (9) is obtained explicitly. In this connection, we note that special class of mixed type of solitary wave solutions of system (1) has been derived recently by following the Hirota approach [17, 18] but these are not strict two-soliton solutions, as are those we have obtained.

In this way, proceeding further one can generalize the expression for $g$, $h$ and $f$ corresponding to the $N$-soliton solutions as

$$g=\sum_{\alpha=0,1} M_1(\alpha)\exp\left(\sum_{j=1}^{2N}\alpha_j\eta_j+\sum_{1\leqslant i<j}^{2N}\alpha_i\alpha_j\phi_{ij}\right) \tag{29a}$$

$$h=\sum_{\alpha=0,1} M_2(\alpha)\exp\left(\sum_{j=1}^{2N}\alpha_j\eta_j+\sum_{1\leqslant i<j}^{2N}\alpha_i\alpha_j\phi_{ij}\right) \tag{29b}$$

$$f=\sum_{\alpha=0,1} M_3(\alpha)\exp\left(\sum_{j=1}^{2N}\alpha_j\eta_j+\sum_{1\leqslant i<j}^{2N}\alpha_i\alpha_j\phi_{ij}\right) \tag{29c}$$

where

$$n_j=k_j(t+ik_jx)+\eta_j^{(0)} \qquad j=1,2,\ldots,2N$$

$$\eta_{j+N}=\eta_j^* \qquad k_{j+N}=k_j^* \qquad \text{for } j=1,2,\ldots,N$$

$$\exp(\phi_{ij})=\frac{\mu[1+\exp(\varepsilon+\varepsilon^*)]}{(k_i+k_j)^2} \qquad \text{for } i=1,2,\ldots,N \quad \text{and} \quad j=N+1,\ldots,2N$$

$$=\frac{(k_i-k_j)^2}{\mu[1+\exp(\varepsilon+\varepsilon^*)]} \qquad \text{for } i=1,2,\ldots,N \quad \text{and} \quad j=1,2,\ldots,N$$

$$i=N+1,\ldots,2N \quad \text{and} \quad j=N+1,\ldots,2N$$

and

$$M_1(\alpha)=\begin{cases}1 & \text{when } 1+\sum_{i=1}^{N}\alpha_{i+N}=\sum_{i=1}^{N}\alpha_i\\ 0 & \text{otherwise}\end{cases}$$

$$M_2(\alpha)=\begin{cases}e^{\varepsilon_0} & \text{when } 1+\sum_{i=1}^{N}\alpha_{i+N}=\sum_{i=1}^{N}\alpha_i\\ 0 & \text{otherwise}\end{cases}$$

$$M_3(\alpha)=\begin{cases}1 & \text{when } \sum_{i=1}^{N}\alpha_{i+N}=\sum_{i=1}^{N}\alpha_i\\ 0 & \text{otherwise}\end{cases} \tag{30}$$

## 3. Dark solitons

Now proceeding to the case of the normal GVD region, equations (1) for the parametric choice (2), where dark solitons occur, can be written as

$$iq_{1x}-q_{1tt}+2\mu(|q_1|^2+|q_2|^2)q_1=0 \tag{31a}$$

$$iq_{2x}-q_{2tt}+2\mu(|q_1|^2+|q_2|^2)q_2=0. \tag{31b}$$

Again using the transformation (8), the system (31) can be rewritten as

$$f[(\mathrm{i}D_x - D_t^2)g\cdot f] + g[D_t^2 f\cdot f + 2\mu(gg^* + hh^*)] = 0 \tag{32a}$$

$$f[(\mathrm{i}D_x - D_t^2)h\cdot f] + h[D_t^2 f\cdot f + 2\mu(gg^* + hh^*)] = 0. \tag{32b}$$

Equations (32) can be decoupled into the set of bilinear equations as

$$\mathscr{B}_1 g\cdot f = 0 \qquad \mathscr{B}_1 h\cdot f = 0 \qquad \mathscr{B}_2 f\cdot f = -2\mu(gg^* + hh^*) \tag{33a}$$

where the bilinear operators $\mathscr{B}_1$ and $\mathscr{B}_2$ are defined as

$$\mathscr{B}_1 = (\mathrm{i}D_x - D_t^2 - \lambda) \qquad \mathscr{B}_2 = (D_t^2 + \lambda) \tag{33b}$$

in which $\lambda$ is constant to be determined.

Next, in order to construct a dark one-soliton solution for the system (31), we assume

$$g = g_0(1 + \chi g_1) \qquad h = h_0(1 + \chi h_1) \qquad f = 1 + \chi f_1. \tag{34}$$

Substituting (34) in (33) and then collecting the coefficients of $\chi^{(0)}$, we get

$$\mathscr{B}_1 g_0\cdot 1 = 0 \qquad \mathscr{B}_1 h_0\cdot 1 = 0 \qquad -2\mu(g_0 g_0^* + h_0 h_0^*) = \lambda. \tag{35}$$

In order to satisfy (35), we consider

$$g_0 = \tau_1 \exp(\mathrm{i}\psi_1) \qquad h_0 = \tau_2 \exp(\mathrm{i}\psi_2) \tag{36a}$$

where

$$\psi_i = l_i t - (\lambda - l_i^2)x + \psi_i^{(0)} \qquad i = 1, 2 \tag{36b}$$

$$|\tau_i|^2 + |\tau_2|^2 = -(\lambda/2\mu) \tag{37}$$

in which $l_1$, $l_2$, $\psi_1^{(0)}$ and $\psi_2^{(0)}$ are real constants and $(\tau_1, \tau_2)$ are complex constants.

Using (36) and the usual Hirota identities [13], the coefficients of $\chi$ lead to

$$\begin{aligned} &\mathscr{C}_1(1\cdot f_1 + g_1\cdot 1) = 0 \qquad \mathscr{C}_2(1\cdot f_1 + h_1\cdot 1) = 0 \\ &\mathscr{B}_2(1\cdot f_1 + f_1\cdot 1) + 2\mu[\tau_1^2(g_1 + g_1^*) + \tau_2^2(h_1 + h_1^*)] = 0 \end{aligned} \tag{38}$$

where

$$\mathscr{C}_j = (\mathrm{i}D_x - 2\mathrm{i}l_j D_t - D_t^2) \qquad j = 1, 2. \tag{39}$$

One can easily check that equations (38) admit the following solutions,

$$g_1 = Z_g \exp(\xi_1) \qquad h_1 = Z_h \exp(\xi_1) \qquad f_1 = \exp(\xi_1) \tag{40}$$

where

$$\xi_1 = P_1 t - \Omega_1 x + \xi_1^{(0)} \tag{41}$$

in which $P_1$, $\Omega_1$ and $\xi_1^{(0)}$, are real constants and $(Z_g, Z_h)$ are complex constants, connected by the relations

$$Z_g = \frac{-[P_1^2 - \mathrm{i}(\Omega_1 + 2l_1P_1)]^2}{P_1^4 + (\Omega_1 + 2l_1P_1)^2} \tag{42}$$

$$Z_h = \frac{-[P_1^2 - \mathrm{i}(\Omega_1 + 2l_2P_1)]^2}{P_1^4 + (\Omega_1 + 2l_2P_1)^2} \tag{43}$$

and

$$\frac{|\tau_1|^2}{P_1^4+(\Omega_1+2l_1P_1)^2}+\frac{|\tau_2|^2}{P_1^4+(\Omega_1+2l_2P_1)^2}=\frac{1}{4\mu P_1^2}. \tag{44}$$

From (42) and (43), it can easily be seen that $|Z_g|^2=|Z_h|^2=1$. Using (44) and (37), the expression for $|\tau_1|^2$ and $|\tau_2|^2$ can be obtained. It can also be checked that the equations corresponding to the coefficients of $\chi^2$ are identically satisfied for the solutions (40). Using (34), after absorbing $\chi$, the dark one-soliton solution can be derived as

$$q_1=-\frac{\tau_1}{2}\exp(\mathrm{i}\psi_1)[(1+Z_g)-(1-Z_g)\tanh(\xi_1/2)], \tag{45a}$$

$$q_2=-\frac{\tau_2}{2}\exp(\mathrm{i}\psi_2)[(1+Z_h)-(1-Z_h)\tanh(\xi_1/2)]. \tag{45b}$$

Thus here we obtain the dark one-soliton solution (45) by following the Hirota technique systematically. We also note that Kivshar and Turitsyn [8] have obtained a special form of the above dark one-soliton solution, using an ansatz, corresponding to the case $U_0=-\mathrm{i}\tau_1\sqrt{Z_g}$, $V_0=-\mathrm{i}\tau_2\sqrt{Z_h}$ being real in equation (45).

For constructing dark two-soliton solutions we now assume

$$g=g_0(1+\chi g_1+\chi^2 g_2) \qquad h=h_0(1+\chi h_1+\chi^2 h_2) \qquad f=1+\chi f_1+\chi^2 f_2 \tag{46}$$

where $g_0$ and $h_0$ are obtained here as in equation (36*a*). Using (36), (46) and the usual Hirota identities [13], we obtain the following set of equations from (33), corresponding to the different powers of $\chi$ as

$\chi^1$: Equations (38)

$$\chi^2:\quad \mathscr{C}_1(1\cdot f_2+g_1\cdot f_1+g_2\cdot 1)=0 \qquad \mathscr{C}_2(1\cdot f_2+h_1\cdot f_1+h_2\cdot 1)=0$$
$$\mathscr{B}_2(1\cdot f_2+f_1\cdot f_1+f_2\cdot 1)+2\mu[\tau_1^2(g_2+g_2^*+g_1g_1^*)+\tau_2^2(h_2+h_2^*+h_1h_1^*)]=0 \tag{47}$$

$$\chi^3:\quad \mathscr{C}_1(g_1\cdot f_2+g_2\cdot f_1)=0 \qquad \mathscr{C}_2(h_1\cdot f_2+h_2\cdot f_1)=0$$
$$\mathscr{B}_2(f_1\cdot f_2+f_2\cdot f_1)+2\mu[\tau_1^2(g_1g_2^*+g_2g_1^*)+\tau_2^2(h_1h_2^*+h_2h_1^*)]=0 \tag{48}$$

$$\chi^4:\quad \mathscr{C}_1 g_2\cdot f_2=0 \qquad \mathscr{C}_2 h_2\cdot f_2=0 \qquad \mathscr{B}_2 f_2\cdot f_2+2\mu(\tau_1^2 g_2g_2^*+\tau_2^2 h_2h_2^*)=0. \tag{49}$$

One can easily find the solutions for the above set of equations as

$$\begin{aligned} &g_1=h_1=Z_1\exp(\xi_1)+Z_2\exp(\xi_2) \qquad f_1=\exp(\xi_1)+\exp(\xi_2)\\ &g_2=h_2=A_{12}Z_1Z_2\exp(\xi_1+\xi_2) \qquad f_2=A_{12}\exp(\xi_1+\xi_2)\end{aligned} \tag{50}$$

where

$$\xi_j=P_jt-\{[4\mu(|\tau_1|^2+|\tau_2|^2)-P_j^2]^{1/2}-2l_1\}P_jz+\xi_j^{(0)} \tag{51}$$

$$Z_j=\frac{-P_j+\mathrm{i}[4\mu(|\tau_1|^2+|\tau_2|^2)-P_j^2]^{1/2}}{P_j+\mathrm{i}[4\mu(|\tau_1|^2+|\tau_2|^2)-P_j^2]^{1/2}} \qquad j=1,2 \tag{52}$$

and

$$A_{12}=\frac{(P_1-P_2)^2+\{[4\mu(|\tau_1|^2+|\tau_2|^2)-P_1^2]^{1/2}-[4\mu(|\tau_1|^2+|\tau_2|^2)-P_2^2]^{1/2}\}^2}{(P_1+P_2)^2+\{[4\mu(|\tau_1|^2+|\tau_2|^2)-P_1^2]^{1/2}-[4\mu(|\tau_1|^2+|\tau_2|^2)-P_2^2]^{1/2}\}^2}. \tag{53}$$

Here the constants $P_j$ and $\xi_j^{(0)}$ are real and we have restricted to the choice $l_1 = l_2$ in (36*a*) so that equations (47)–(49) are consistent. Now using (36), (46) and (50) in (8), the dark two-solitons can be found explicitly.

In this way, by proceeding further the dark *N*-soliton solutions can be derived using the following equations in (8):

$$g = \tau_1 \exp(\mathrm{i}\psi_1)\left\{\sum_{\alpha=0,1} \exp\left[\sum_{j=1}^{N} \alpha_j(\xi_j + \mathrm{i}\theta_j) + \sum_{i<j}^{N} a_{ij}\alpha_i\alpha_j\right]\right\}$$

$$h = \tau_2 \exp(\mathrm{i}\psi_1)\left\{\sum_{\alpha=0,1} \exp\left[\sum_{j=1}^{N} \alpha_j(\xi_j + \mathrm{i}\theta_j) + \sum_{i<j}^{N} a_{ij}\alpha_i\alpha_j\right]\right\}$$

$$f = \sum_{\alpha=0,1} \exp\left[\left(\sum_{j=1}^{N} \alpha_j\xi_j + \sum_{i<j}^{N} a_{ij}\alpha_i\alpha_j\right)\right] \qquad (54)$$

where

$$\xi_j = P_j t - \{[4\mu(|\tau_1|^2 + |\tau_2|^2) - P_j^2]^{1/2} - 2l_1\}P_j z + \xi_j^{(0)}$$

$$\exp(\mathrm{i}\theta_j) = \frac{-P_j + \mathrm{i}[4\mu(|\tau_1|^2 + |\tau_2|^2) - P_j^2]^{1/2}}{P_j + \mathrm{i}[4\mu(|\tau_1|^2 + |\tau_2|^2) - P_j^2]^{1/2}}$$

and

$$\exp(a_{ij}) = \frac{(P_i - P_j)^2 + \{[4\mu(|\tau_1|^2 + |\tau_2|^2) - P_i^2]^{1/2} - [4\mu(|\tau_1|^2 + |\tau_2|^2) - P_j^2]^{1/2}\}^2}{(P_i + P_j)^2 + \{[4\mu(|\tau_1|^2 + |\tau_2|^2) - P_i^2]^{1/2} - [4\mu(|\tau_1|^2 + |\tau_2|^2) - P_j^2]^{1/2}\}^2}$$

$$\text{for } i, j = 1, 2, \ldots, N \qquad (55)$$

Here $\tau_1$ and $\tau_2$ are again related to the separation constant $\lambda$, as in (37).

Finally, we show that for the second parametric choice (3) too, one can obtain dark solitons of the form (45). For this case, assuming $c_1 = -c_2 = -c$ (say), $\alpha = -\beta = \gamma = \delta$ (say), the system (1) becomes

$$\mathrm{i}q_{1x} - cq_{1tt} + 2\delta(|q_1|^2 - |q_2|^2)q_1 = 0 \qquad (56a)$$

$$\mathrm{i}q_{2x} + cq_{2tt} - 2\delta(|q_1|^2 - |q_2|^2)q_2 = 0 \qquad (56b)$$

Substituting again transformation (8) in (56) and then decoupling the resultant equations, we obtain the bilinear equations

$$\mathscr{E}_1 g \cdot f = 0 \qquad \mathscr{E}_2 h \cdot f = 0 \qquad \mathscr{E}_3 f \cdot f = -(2\delta/c)(gg^* - hh^*) \qquad (57)$$

where

$$\mathscr{E}_1 = (\mathrm{i}D_x - cD_t^2 - \Gamma) \qquad \mathscr{E}_2 = (\mathrm{i}D_x + cD_t^2 + \Gamma) \qquad \mathscr{E}_3 = (D_t^2 + (\Gamma/c)) \qquad (58)$$

in which $\Gamma$ is a constant to be determined.

In order to find the one-soliton solution, we make the same ansatz (34) in (57) and then, proceeding as in the previous case, we obtain

$$q_1 = -\frac{\Delta_1}{2}\exp(\mathrm{i}\phi_1)[(1 + Z_1) - (1 - Z_1)\tanh(\zeta_1/2)] \qquad (59a)$$

$$q_2 = -\frac{\Delta_2}{2}\exp(-\mathrm{i}\phi_2)[(1 + Z_2) - (1 - Z_2)\tanh(\zeta_1/2)] \qquad (59b)$$

where

$$\phi_j = m_j t - (\Gamma - c m_j^2)x + \phi_j^{(0)} \qquad j = 1, 2 \tag{60}$$

$$\zeta_1 = p_1 t - \omega_1 x + \zeta_1^{(0)} \tag{61}$$

$$Z_1 = \frac{-[c p_1^2 - \mathrm{i}(\omega_1 + 2c m_1 p_1)]^2}{c^2 p_1^4 + (\omega_1 + 2c m_1 p_1)^2} \qquad Z_2 = \frac{-[c p_1^2 + \mathrm{i}(\omega_1 + 2c m_2 p_1)]^2}{c^2 p_1^4 + (\omega_1 + 2c m_2 p_1)^2}. \tag{62}$$

Here $m_1, m_2, p_1, \omega_1, \phi_1^{(0)}, \phi_2^{(0)}$ and $\zeta_1^{(0)}$ are all real constants and $(\Delta_1, \Delta_2)$ are complex constants, related by the relations

$$|\Delta_1|^2 - |\Delta_2|^2 = -(\Gamma/2\delta) \tag{63}$$

and

$$\frac{|\Delta_1|^2}{c^2 p_1^4 + (\omega_1 + 2c m_1 p_1)^2} - \frac{|\Delta_2|^2}{c^2 p_1^4 + (\omega_1 + 2c m_2 p_1)^2} = \frac{1}{4\delta c p_1^2}. \tag{64}$$

By proceeding further, as in the previous case, higher-order soliton solutions can also be given explicitly. To our knowledge the explicit soliton solutions of the system (56) have not been reported previously.

## 4. Conclusion

In conclusion, we have explicitly obtained bright and dark $N$-solitons for the integrable cases of coupled NLS equations (1) describing two-mode propagation in optical fibre, using the Hirota method, derivable from P-analysis. The form of the bright one-soliton agrees with the result derivable from the inverse scattering analysis [3], and a special case (corresponding to a specific choice of the parameters) of our dark one-soliton agrees with that obtained by an *ad hoc* method [8]. We have also reported the explicit soliton solutions of the systems (1) corresponding to the parametric choice (3).

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